7-4-2016

Exact solution for viscous fluid flow in porous medium with magnetic field

Dinesh P. A.
M.S.Ramaiah Institute of Technology, dineshdpa@hotmail.com

Jayalakshmamma D.V.
M.S.Ramaiah Institute of Technology

Chandrashekhar D.V.
M.S.Ramaiah Institute of Technology

EXACT SOLUTION FOR VISCOUS FLUID FLOW IN POROUS MEDIA WITH MAGNETIC FIELD

Dinesh P A
Department of Mathematics, M. S. Ramaiah Institute of Technology, Bangalore –560 054, Karnataka,India
E-mail: dineshdpa@hotmail.com

Jayalakshmamma D V
Department of Mathematics, Vemana Institute of Technology, Bangalore –560 034, Karnataka, India

Chandrashekhar D V
Department of Mathematics, Vivekananda Institute of Technology, Bangalore – 560 074, Karnataka, India

ABSTRACT
The steady flow of viscous, incompressible, conducting fluid flow past a spherical solid core embedded in another spherical porous medium is considered. The exact solution is obtained for the flow in the presence of transverse magnetic field. The considered fluid flow is governed by Brinkman equation in porous region and by Stokes equation in the fluid region with additional Lorentz's force due to applied magnetic field. The flows in the two regions are matched across the interface by assuming continuity of velocity and stress across the interface. Further, no-slip condition at the solid surface and uniform velocity far from the flow region are used. The solutions are obtained by similarity transformation method in terms of modified Bessel's functions. The expression for tangential shear stress, normal and tangential velocity is obtained. The results are demonstrated by graphs for various non-dimensional parameters. It is noticed that diffusion of the fluid into porous region is more as magnetic field strength is amplified. This shows the suppression of the flow in the presence of magnetic field. Also, the amplitude of the shearing stress intensifies with increase in the magnetic field strength and lessens with raise in porous parameter.

KEYWORDS: Lorentz's force, Uniform velocity, Magnetic field, Bessel function, shear stress.

1. INTRODUCTION
The study of hydrodynamic flows in presence of magnetic field has attracted many authors due to vast applications in astrophysical, geophysical and industrial fields. Many practical problems need a mechanism to control the motion of the fluid past solid bodies with Magnetohydrodynamics (MHD) effects. The study of magnetohydrodynamic flows of electrically conducting fluids in electric and magnetic fields is of considerable interest in modern metallurgical and metal working processes. This has led to considerable interest in the study of boundary layer flows subjected to an externally applied magnetic field.

Anjali Devi and Raghavachar [1982] studied the horizontal flow of a vertically stratified, electrically conducting fluid past a non conducting sphere in the presence of uniform magnetic field for non-diffusive medium. Kyrlidis et al [1990] presented the study of conducting fluid past axi-symmetric bodies in the presence of magnetic field in the limit of small inertial and magnetic Reynolds numbers. Chandran et al [1996] have been analyzed the effect of magnetic field on the flow heat transfer past a continuously moving porous plate in a stationary fluid. The steady, viscous, electrically conducting fluid flow around a circular cylinder in the presence of magnetic field applied parallel to the main flow was investigated by Raghava Rao and Sekhar [2000]. Finite difference method was used to solve the non-linear Navier-Stokes equation. Jayalakshmamma et al [2011] presented a creeping flow past a composite sphere in presence of magnetic field. matching boundary conditions are applied at the interface of the fluid and porous media. Pal and Talukdar [2011] analyzed an investigation on the unsteady flow of a laminar two-dimensional oscillatory flow of an incompressible electrically conducting viscous fluid between two non-conducting parallel plane surfaces in the presence of suction / injection. Recently flow of conducting fluid on solid core surrounded by porous cylindrical region in presence of transverse magnetic field is presented by Jayalakshmamma et al [2014].

The control of fluid flow during the relative motion of a body or fluid under the influence of magnetic field to is the main objective of this article. Hence, in this paper, the Stokes flow of a steady, incompressible, viscous, electrically conducting fluid flow past a solid sphere embedded in a spherical porous medium is presented in the presence of transverse magnetic field. The analytical method is given to find an exact solution for the considered flow.

2. MATHEMATICAL FORMULATION

The two-dimensional flow of a steady, incompressible, viscous, electrically conducting fluid past a stationary solid sphere has been considered in presence of transverse magnetic field. A fixed solid sphere of radius a is surrounded by a spherical porous media of radius b $(a<b)$. Further, it is also assumed that the induced magnetic field is negligible in comparison with the applied magnetic field. The domain has been divided into two regions namely, porous and fluid regions. The governing equations which describe the flow of a conducting fluid in fluid region can be written as:

$$\nabla\cdot\vec{q}_1=0, \tag{1}$$

$$\nabla p_1=\mu\nabla^2\vec{q}_1+\mu_h^2\sigma_e\left(\vec{q}_1\times\vec{H}\right)\times\vec{H}, \tag{2}$$

where $\vec{q}_1=(u_1,v_1,w_1)$ is the velocity in the fluid region, μ is the viscosity of the fluid, μ_h is the magnetic permeability, σ_e is the electrical conductivity, which is very small so that the induced magnetic field is negligible, $\vec{H}$ is the uniform magnetic field and p_1 is the hydrostatic pressure of the fluid region. Here equation (2) is said to be modified Stokes equation as it consist of Lorentz force due to applied Magnetic field, along with the viscous term on the right hand side of the equation.

The flow in the porous region $a<r\le b$ is governed by the modified Brinkman equation along with equation of continuity by:

$$\nabla\cdot\vec{q}_2=0, \tag{3}$$

$$\nabla p_2=\bar{\mu}\nabla^2\vec{q}_2-\frac{\mu}{k}\vec{q}_2+\mu_h^2\sigma_e\left(\vec{q}_2\times\vec{H}\right)\times\vec{H}, \tag{4}$$

where, $\vec{q}_2=(u_2,v_2,w_2)$ is the velocity in the porous region, $\bar{\mu}$ is the Brinkman viscosity, p_2 the hydrostatic pressure of the porous region and k the permeability of the porous region.

Spherical co-ordinate system (r,θ,ϕ) with the origin at the center of the sphere has been used and the axis $\theta=0$ is chosen along the direction of the uniform velocity u_∞ far from the fluid region. Also due to axi-symmetry, we have $\frac{\partial}{\partial\phi}=0$. The flow characteristics of the problem are described by equations (1) to (4) can be analyzed in terms of non-dimensional parameters defined as:

$$r^*=\frac{r}{a},\ \vec{q}_1^{\,*}=\frac{\vec{q}_1}{u_\infty},\ \vec{q}_2^{\,*}=\frac{\vec{q}_2}{u_\infty},\ \vec{H}_1^{*}=\frac{\vec{H}_1}{H_0},\ p_1^*=\frac{ap_1}{\mu u_\infty},\ p_2^*=\frac{ap_2}{\mu u_\infty} \tag{5}$$

where H_0 is the applied constant magnetic field.

Using the dimensionless variables from equation (5), the governing equations (1) and (2), for spherical polar co-ordinate system can be written as

$$\frac{\partial}{\partial r}\left(r^2u_1\right)+\frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_1\sin\theta\right)=0, \tag{6}$$

$$\frac{\partial p_1}{\partial r}=\begin{bmatrix}\frac{\partial^2u_1}{\partial r^2}+\frac{2}{r}\frac{\partial u_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2u_1}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_1}{\partial\theta}\\ -\frac{2u_1}{r^2}-\frac{2}{r^2}\frac{\partial v_1}{\partial r}-\frac{2v_1\cot\theta}{r^2}\end{bmatrix}-M^2u_1 \tag{7}$$

$$\frac{1}{r}\frac{\partial p_1}{\partial\theta}=\begin{bmatrix}\frac{\partial^2v_1}{\partial r^2}+\frac{2}{r}\frac{\partial v_1}{\partial r}+\frac{1}{r^2}\frac{\partial^2v_1}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_1}{\partial\theta}\\ +\frac{2}{r^2}\frac{\partial u_1}{\partial\theta}-\frac{v_1\operatorname{cosec}^2\theta}{r^2}\end{bmatrix}-M^2v_1 \tag{8}$$

where $M=\sqrt{\frac{\mu_h^2\sigma_e a^2H_0^2}{\mu}}$ is the Hartmann number.

Similarly, using the transformation from equation (5), the equations (3) and (4) are non-dimensionalised for the porous region, and the corresponding equations in spherical polar co-ordinate system can be written as:

$$\frac{\partial}{\partial r}\left(r^2u_2\right)+\frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(v_2\sin\theta\right)=0, \tag{9}$$

$$\frac{\partial p_2}{\partial r}=\begin{bmatrix}\frac{\partial^2 u_2}{\partial r^2}+\frac{2}{r}\frac{\partial u_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u_2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u_2}{\partial \theta}\\ -\frac{2u_2}{r^2}-\frac{2}{r^2}\frac{\partial v_2}{\partial r}-\frac{2v_2\cot\theta}{r^2}\end{bmatrix}-S^2u_2 \qquad (10)$$

$$\frac{1}{r}\frac{\partial p_2}{\partial \theta}=\begin{bmatrix}\frac{\partial^2 v_2}{\partial r^2}+\frac{2}{r}\frac{\partial v_2}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_2}{\partial \theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_2}{\partial \theta}\\ +\frac{2}{r^2}\frac{\partial u_2}{\partial \theta}-\frac{v_2\cos ec^2\theta}{r^2}\end{bmatrix}-S^2v_2 \qquad (11)$$

where $S^2=\sigma^2+M^2$, in which $\sigma=\frac{a}{\sqrt{k}}$ is the porous parameter.

Since flow is axi-symmetric, the stream function $\psi_i(r,\theta)$ (where $i=1,2$ corresponds to fluid and porous regions respectively) is introduced, such that the equation of continuity is satisfied in spherical polar co-ordinate system for both fluid and porous regions respectively. It is defined as follows:

$$u_i=\frac{1}{r^2\sin\theta}\frac{\partial \psi_i}{\partial \theta};\;\; v_i=\frac{-1}{r\sin\theta}\frac{\partial \psi_i}{\partial r}. \qquad (12)$$

By eliminating the pressure term from equations (7), and (8) of fluid region and equations (10), and (11) of porous region by cross differentiation a fourth order linear partial differential equation in terms of stream function is obtained as:

$$E^4\psi_1-M^2E^2\psi_1=0, \quad b\le r<\infty, \qquad (13)$$

$$E^4\psi_2-S^2E^2\psi_2=0, \quad a\le r<b, \qquad (14)$$

where $E^2=\frac{\partial^2}{\partial r^2}+\frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\right)$ is the Laplacian operator in spherical co-ordinate system.

To determine the flow field it is necessary to apply the boundary conditions. The different types of interfacial boundary conditions have been postulated to describe flow characteristics at the boundary between a porous and fluid medium or between a porous medium and solid surface. The physically realistic and mathematically consistent boundary conditions, which describe the present problem, are no slip condition at the solid surface, continuity of velocity components, continuity of normal and tangential stress components at the interface of the porous and fluid regions and the velocity becomes uniform far away from the fluid region. The no slip condition at the surface of the solid surface for the present problem takes the form:

$$u_2(a,\theta)=0, \quad 0\le\theta\le 2\pi, \qquad (15)$$

$$v_2(a,\theta)=0, \quad 0\le\theta\le 2\pi. \qquad (16)$$

The interfacial conditions, continuity of normal and tangential velocity components, continuity of normal and tangential stress components at the interface of the porous and fluid region are given by:

$$u_2(b,\theta)=u_1(b,\theta), \quad 0\le\theta\le 2\pi, \qquad (17)$$

$$v_2(b,\theta)=v_1(b,\theta), \quad 0\le\theta\le 2\pi, \qquad (18)$$

$$\tau_{r\theta(2)}(b,\theta)=\tau_{r\theta(1)}(b,\theta), \quad 0\le\theta\le 2\pi, \qquad (19)$$

$$\tau_{rr(2)}(b,\theta)=\tau_{rr(1)}(b,\theta), \quad 0\le\theta\le 2\pi, \qquad (20)$$

where $\tau_{r\theta(1)}$ and $\tau_{rr(1)}$ represents the dimensionless tangential and normal components of stress tensors in the porous region and is given by:

$$\tau_{r\theta(1)}=\frac{1}{r}\frac{\partial u_1}{\partial \theta}+\frac{\partial v_1}{\partial r}-\frac{v_1}{r},$$

(21)

$$\tau_{rr(1)}=-p_1+2\frac{\partial u_1}{\partial r}. \qquad (22)$$

Further, $\tau_{r\theta(2)}$ and $\tau_{rr(2)}$ stands for dimensionless tangential and normal components stress tensors in the fluid region, and are also defined in the same way as equations (21) and (22).The continuity of the normal stress at the interface of the two regions in equation (20), shows the continuity of pressure across the interface, since it was assumed that the viscosity of the fluid is equal to the Brinkman viscosity $\bar{\mu}=\mu$. Therefore, equation (20) reduces to:

$$p_2(b,\theta)=p_1(b,\theta), \quad 0\le\theta\le 2\pi. \qquad (23)$$

Further, the stream function in the fluid region far away from the boundary is given by:

$$\psi_1(r,\theta)=\frac{1}{2}\left(r^2-\frac{1}{r}\right)\sin^2\theta, \quad b\le r<\infty. \qquad (24)$$

From equation (12) and equation (24), the boundary conditions for the velocity components far from the fluid region are:

$$u_1\sim\cos\theta, \;\; v_1\sim -\sin\theta' \text{ as } r\to\infty \qquad (25)$$

Hence the boundary condition far away from the fluid region, equation (24) reduces in terms of stream function as:

$$\psi_1(r,\theta) \sim \frac{r^2}{2}\sin^2\theta, \text{ as } \quad r \to \infty. \tag{26}$$

3. METHOD OF SOLUTION

The boundary condition from equation (26) suggests the following similarity solution to equation (13) and (14) as:

$$\psi_1(r,\theta) = f_1(r)\sin^2\theta, \quad b \le r < \infty, \tag{27}$$

$$\psi_2(r,\theta) = f_2(r)\sin^2\theta, \quad a \le r < b. \tag{28}$$

Substituting equation (27) in (13) and equation (28) in (14), the fourth order partial differential equation in terms of stream functions $\psi_1(r,\theta)$ and $\psi_2(r,\theta)$ reduces to fourth order ordinary differential equation in $f_1(r)$ and $f_2(r)$ respectively, and are in the following form:

$$\begin{aligned} & f_1^{iv}(r) - \frac{4}{r^2} f_1''(r) + \frac{8}{r^3} f_1'(r) - \frac{8}{r^4} f_1(r) \\ & - M^2\left(f_1''(r) - \frac{2}{r^2} f_1(r)\right) = 0, \; b \le r < \infty \end{aligned} \tag{29}$$

$$\begin{aligned} & f_2^{iv}(r) - \frac{4}{r^2} f_2''(r) + \frac{8}{r^3} f_2'(r) - \frac{8}{r^4} f_2(r) \\ & - S^2\left(f_2''(r) - \frac{2}{r^2} f_2(r)\right) = 0, \; a \le r < b \end{aligned} \tag{30}$$

The corresponding boundary conditions in terms of $f_1(r)$ and $f_2(r)$ from equations (15) to (20) and from equation (26) are as follows. No-slip condition at the surface of the solid sphere is given by:

$$f_2(a) = 0, \tag{31}$$

$$f_2'(a) = 0. \tag{32}$$

The matching condition at the interface of the porous and fluid region takes the form:

$$f_2(b) = f_1(b), \tag{33}$$

$$f_2'(b) = f_1'(b), \tag{34}$$

$$f_2''(b) = f_1''(b), \tag{35}$$

$$f_2'''(b) - \sigma^2 f_2'(b) = f_1'''(b). \tag{36}$$

Further, the uniform velocity far away from the boundary, from equation (26) reduces to:

$$f_1(r) \sim \frac{r^2}{2} \quad \text{as } \; r \to \infty. \tag{37}$$

The solution for the equations (29) and (30) is obtained analytically by similarity solution method and obtained solution is

$$f_1(r) = \frac{A_1}{r^2} + B_1 r^2 + C_1\sqrt{r}\, I_{3/2}(Mr) + D_1\sqrt{r}\, K_{3/2}(Mr), \; b \le r < \infty \tag{38}$$

$$f_2(r) = \frac{A_2}{r^2} + B_2 r^2 + C_2\sqrt{r}\, I_{3/2}(Sr) + D_2\sqrt{r}\, K_{3/2}(Sr), \; a \le r < b \tag{39}$$

For the flow in fluid region as $r \to \infty$ the $I_{3/2}(Mr) \to \infty$. Therefore, the solution for fluid region equation (38) is valid if and only if $C_1 = 0$. Also from the boundary condition, for far away from the fluid region from equation (37) we get $B_1 = 1/2$. Hence equation (46) reduces to:

$$f_1(r) = \frac{A_1}{r^2} + \frac{r^2}{2} + D_1\sqrt{r}\, K_{3/2}(Mr), \; b \le r < \infty, \tag{40}$$

Also, equation (39) for $a \le r < b$ can be written as:

$$\begin{aligned} f_2(r) = & \frac{A_2}{r^2} + B_2 r^2 + C_2\left(\frac{\cosh(Sr)}{Sr} - \sinh(Sr)\right) \\ & + D_2\left(\frac{\sinh(Sr)}{Sr} - \cosh(Sr)\right) \end{aligned} \tag{41}$$

Here the arbitrary constants A_1, D_1, A_2, B_2, C_2 and D_2 are determined using the boundary conditions mentioned above from equations (31) to (37).

4. RESULTS AND DISCUSSION

The flow of steady incompressible viscous and electrically conducting fluid past a solid sphere of radius a embedded in a porous sphere of radius b $(a < b)$, has been investigated in presence of an uniform magnetic field, applied in the transverse direction of the fluid motion. The modified Stokes and Brinkman equation are used to describe the flow in fluid and porous regions respectively. The analytical solution is obtained by similarity solution method. The continuity of velocity, tangential and normal stress are used as the interface boundary conditions between fluid and porous regions. The no-slip condition at the surface of the solid sphere and uniform velocity away from the porous sphere are considered. The variation of the stream functions in the boundary layer for different values of the dimensionless parameters are presented. Further, the variation of amplitude of the shearing stress is discussed for various porous parameter and Hartmann number.

First, the porous parameter is fixed at a value $\sigma = 5$ and the Hartman number M is varied. For $M = 1$, it is noticed that due to less permeability the amount of fluid flow in the porous region is less and streamlines are far from the solid sphere as observed in Fig. 1(a). For the same value of $\sigma = 5$, as the magnetic field strength is increased to, it is observed that the amount of fluid penetrating into the porous region is more and as a result streamlines are closer and denser around the solid core, as illustrated in Figs. 1 (b), (c) and (d).

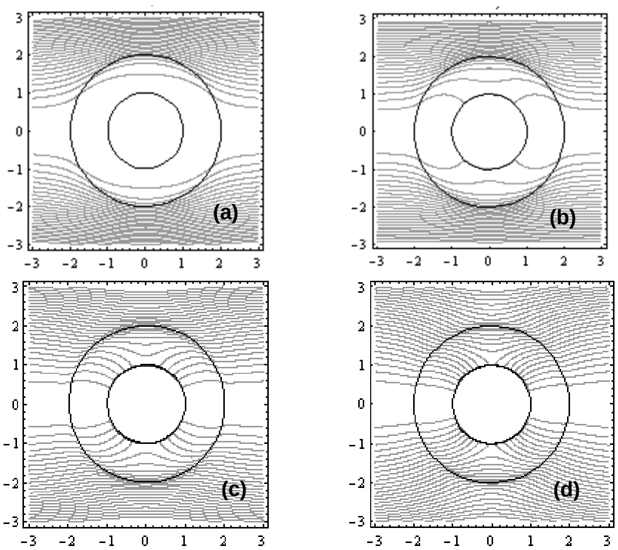


Fig.1 Streamlines for different values of M with $\sigma = 5$
(a) $M = 1$ (b) $M = 2$ c) $M = 5$ (d) $M = 10$

Figures 2(a)-(d) represents the streamlines for different values of porous parameter, σ by fixing the Hartmann number M. The flow behaviour is similar to that of creeping flow of viscous fluid. This can be attributed to a small Hartmann number chosen in the study. Further, as the porous parameter increses, the fluid flow in the porous and fluid region is supressed due to the drag force caused by low permeability. These results are qualitatively consistent with that of Maslyiah et al [1987], where the effect of porous parameter was discussed on fluid flow over a solid core with porous shell.

For the practical importance, the expression for the dimensionless tangential shear stress $\tau_{r\theta(1)}$, at any point on the surface of the solid sphere (i.e., r = 1) is obtained.

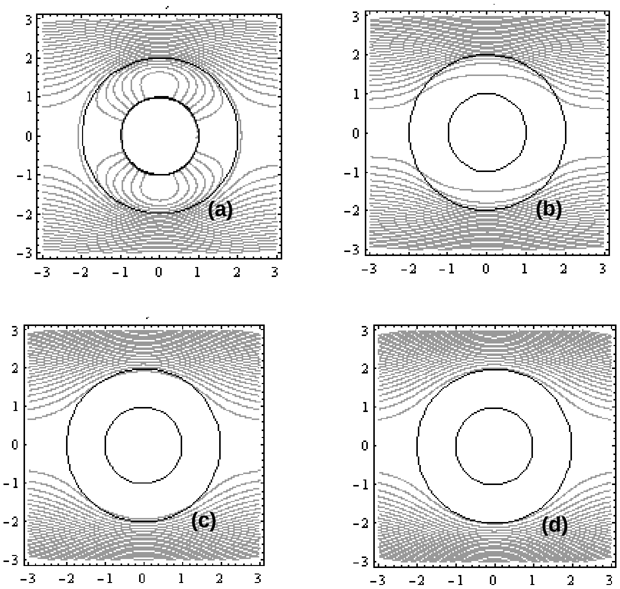
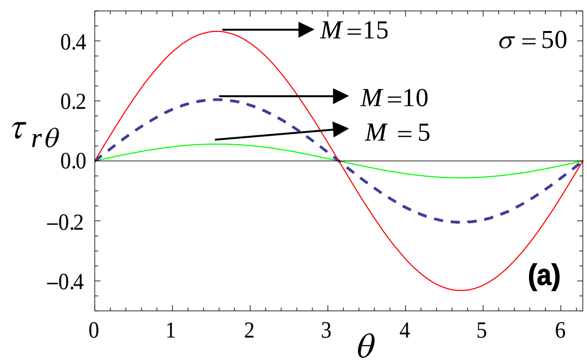


Fig. 2 Streamlines for different values of σ with $M = 1$
(a) $\sigma = 1$ (b) $\sigma = 5$ (c) $\sigma = 10$ (d) $\sigma = 15$



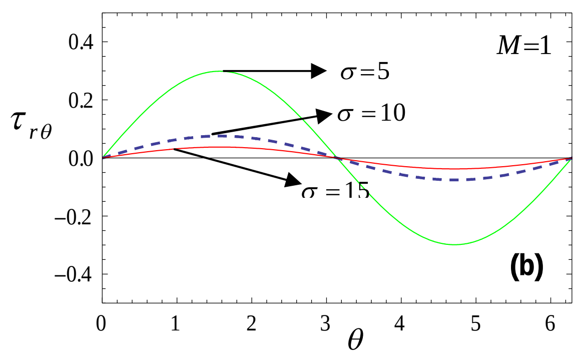


Fig. 3(a) Variation of shear stress for $\sigma = 50$ with $M = 5, 10, 15$ (b) Variation of shear stress for $M = 1$ and $\sigma = 5, 10, 15$

The variation of tangential shear stress $\tau_{r\theta(1)}$ on the surface of the solid sphere for variation of θ is studied for different values of Hartmann numbers and porous parameters and depicted in Figs. 3. (a) and (b). It is observed that the shear stress is periodic in nature, and is also seen at the front and rear points of the sphere for $\theta = 0$ and $\theta = \pi$. However, the shearing stress vanishes and attains its maximum value for $\theta = \pm\frac{\pi}{2}$. Further, it is noticed that the amplitude of the shearing stress amplifies with increase in the magnetic field strength and reduces with increase in porous parameter.

5. CONCLUSIONS

The analytical solution of steady flow of an incompressible viscous and electrically conducting fluid past a solid sphere placed in a spherical porous medium, in the presence of transverse magnetic field. The influence of Hartman number and porous parameter are discussed on the streamline patterns and shearing stress.

The increase in magnetic field strength for a fixed or negligible porous parameter, the meandering of streamlines near the surface of the solid sphere is observed. But, for fixed or negligible Hartman number, the streamlines are moved away from the surface with an increase in porous parameter.

The amplitude of the shearing stress intensifies with increase in the magnetic field strength and lessens with raise in porous parameter.

ACKNOWLEDGEMENT

The authors are grateful to research centre M S Ramaiah Institute of Technology, Vemana Institute of Technology, and Technology and Vivekananda Institute of Technology, Bangalore, India for their support and encouragement to carry out our research work.

REFERENCES

[1] ANJALI DEVI SP,RAGHAVACHAR MR (1982) Magneto hydrodynamic stratified flow past a sphere. Int. J. Engng Sci., 20(10):1169-1177.

[2] KYRLIDIS A, BROWN RA, WALKER JS (1990) Creeping flow of a conducting fluid past axisymmetric bodies in the presence of an aligned magnetic field. Phys. Fluids. A2:2230-228.

[3] CHANDRAN P, SACHETI NC, SINGH AK (1996) Hydromagnetic flow and heat transfer past a continuously moving porous boundary. Int. Comm. Heat Mass Trans. 23 (6):889-898.

[3] RAGHAVA RAO CV, SEKHAR TVS (2000) MHD flow past a circular cylinder-a numerical study. Computational Mechanics. 26:430-436.

[4] JAYALAKSHMAMMA DV, DINESH PA, SANKAR M (2011) Analytical study of creeping flow past a composite sphere: solid core with porous shell in presence of magnetic field. Mapana Journal Science. 10 (2):11-24.

[5] PAL D, TALUKDAR B (2011) Unsteady hydromagnetic oscillating flow past a porous medium with suction/injection and slip effects. Int. J. Appl. Math. Mech. 7(15):58-71.

[6] JAYALAKSHMAMMA DV, DINESH PA, CHANDRASHEKHAR DV (2014) Flow of conducting fluid on solid core surrounded by porous cylindrical region in presence of transverse magnetic field. Mapana Journal Science. 13 (3):13-29.

[7] MASLIYAH JH, NEALE G, MALYSA K, VAN DE VEN TGM (1987) Creeping flow over a composite sphere: Solid core with porous shell. Chem. Engng. Sci. 42 (2):245-253.